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On an Extension of a Result of S. N. Bernstein[1]

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1. INTRODUCTION

Let $f(x)$ be a real-valued continuous function defined on $[-1,+1]$, and, as usual, let

$$E_n(f) \equiv \inf_{p \in \pi_n} \|f-p\|_{L^\infty[-1,+1]}, \qquad \text{for } n = 0,1,2,\ldots, \tag{1}$$

denote the minimum error in the Chebyshev approximation of $f(x)$ over the set π_n of real polynomials of degree at most n. Bernstein [*1*, p. 118] proved that

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0 \tag{2}$$

if and only if $f(x)$ has an analytic extension $f(z)$ such that $f(z)$ is an entire function, i.e., $f(z)$ is analytic for all complex z. Unfortunately, for $f(z)$ entire, (2) does not give any clue as to the *rate* at which $E_n^{1/n}(f)$ tends to zero. One naturally expects that this rate is dependent on the order of $f(z)$, i.e., if $M_f(r) \equiv \max_{|z|\leqslant r} |f(z)|$, then $f(z)$ is said [*2*, p. 8] to be an entire function of *order* σ $(0 \leqslant \sigma < \infty)$ if

$$\overline{\lim_{r\to\infty}} \frac{\ln(\ln M_f(r))}{\ln r} = \sigma. \tag{3}$$

The object of this note is to obtain a sharper form of (2) which depends on the order of f.

2. MAIN RESULT

We now prove

THEOREM 1. *Let $f(x)$ be a real-valued continuous function on* $[-1,+1]$. *Then,*

$$\overline{\lim_{n\to\infty}} \left\{ \frac{n \ln n}{-\ln E_n(f)} \right\} = \sigma \tag{4}$$

where σ is a nonnegative real number if and only if $f(x)$ has an analytic extension $f(z)$ such that $f(z)$ is an entire function of order σ.

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Proof. First, assume that $f(x)$ has an analytic extension $f(z)$ which is an entire function of order σ where $0 \leqslant \sigma < \infty$. Following Bernstein's original proof (cf. [*3*, p. 76] and [*4*, p. 84]), it follows for each $n \geqslant 0$ that

$$E_n(f) \leqslant \frac{2B(\rho)}{\rho^n(\rho - 1)} \qquad \text{for any } \rho > 1, \tag{5}$$

where $B(\rho) \equiv \max_{z \in \mathscr{E}_\rho} |f(z)|$, and $\mathscr{E}_\rho$ with $\rho > 1$ denotes the closed interior of the ellipse with foci ± 1, with half-major axis $(\rho^2 + 1)/(2\rho)$ and half-minor axis $(\rho^2 - 1)/(2\rho)$. The closed disks $D_1(\rho)$ and $D_2(\rho)$ bound the ellipse $\mathscr{E}_\rho$ in the sense that

$$D_1(\rho) \equiv \left\{ z \,\middle|\, |z| \leqslant \frac{\rho^2 - 1}{2\rho} \right\} \subset \mathscr{E}_\rho \subset D_2(p) \equiv \left\{ z \,\middle|\, |z| \leqslant \frac{\rho^2 + 1}{2\rho} \right\}.$$

From this inclusion, it follows by definition that

$$M_f\left(\frac{\rho^2 - 1}{2\rho}\right) \leqslant B(\rho) \leqslant M_f\left(\frac{\rho^2 + 1}{2\rho}\right) \qquad \text{for all } \rho > 1. \tag{6}$$

Consequently, from (5) we have for each $n \geqslant 0$ that

$$E_n(f) \leqslant \frac{2M_f((\rho^2 + 1)/2\rho)}{\rho^n(\rho - 1)} \qquad \text{for any } \rho > 1. \tag{7}$$

Since $f(z)$ is, by assumption, of order σ, then, given any $\epsilon > 0$, there exists an $R(\epsilon) > 0$ such that $M_f(r) < \exp(r^{(\sigma+\epsilon)})$ for all $r \geqslant R(\epsilon)$. Thus,

$$E_n(f) \leqslant \frac{2\exp\{((\rho^2 + 1)/2\rho)^{\sigma+\epsilon}\}}{\rho^n(\rho - 1)} \qquad \text{for all } \rho \geqslant 2R(\epsilon) \text{ and all } n \geqslant 0. \tag{8}$$

The right side of this inequality, considered as a function of ρ for fixed n, is approximately minimized by choosing $\rho = 2n^{1/(\sigma+\epsilon)}$, and this choice of ρ is compatible with the restriction $\rho \geqslant 2R(\epsilon)$ for all n sufficiently large. For this choice of ρ, it is easily verified from (8) that

$$\overline{\lim_{n\to\infty}} \left\{ \frac{n \ln n}{-\ln E_n(f)} \right\} \leqslant \sigma + \epsilon.$$

As ϵ is arbitrary, we thus have that $f(z)$ being of order σ implies that there exists a finite $\beta \geqslant 0$ such that

$$\overline{\lim_{n\to\infty}} \left\{ \frac{n \ln n}{-\ln E_n(f)} \right\} = \beta \leqslant \sigma. \tag{9}$$

We now utilize the relation of (9). From (9), it follows that, given any $\epsilon > 0$, there exists an $n_0(\epsilon) > 0$ such that

$$E_n(f) \leqslant \frac{1}{n^{n/(\beta+\epsilon)}} \qquad \text{for all } n \geqslant n_0(\epsilon), \tag{9'}$$

and from Bernstein's result of (2), this means that $f(x)$ can be extended to an entire function $f(z)$. For each $n \geqslant 0$, there exists a unique polynomial $p_n(x) \in \pi_n$ such that

$$\|f - p_n\|_{L^\infty[-1,+1]} = E_n(f), \qquad n = 0, 1, 2, \ldots.$$

Since $\|p_{n+1} - p_n\|_{L^\infty[-1,+1]}$ is, by the triangle inequality, bounded above by $2E_n(f)$, then another result of Bernstein [*1*, p. 112] (cf. [*3*, p. 42], [*4*, p. 85]) gives us that

$$|p_{n+1}(z) - p_n(z)| \leqslant 2E_n(f)\cdot\rho^{n+1} \qquad \text{for all } z \in \mathscr{E}_\rho \text{ for any } \rho > 1. \tag{10}$$

From this, it follows that we can write

$$f(z) = p_0(z) + \sum_{k=0}^{\infty} (p_{k+1}(z) - p_k(z)),$$

and this series converges uniformly in any bounded domain of the complex plane. Thus, from (10),

$$|f(z)| \leqslant |p_0(z)| + 2\sum_{k=0}^{\infty} E_k(f)\,\rho^{k+1} \qquad \text{for any } z \in \mathscr{E}_\rho, \tag{11}$$

and consequently, from the definition of $B(\rho)$,

$$B(\rho) \leqslant |p_0(z)| + 2\sum_{k=0}^{\infty} E_k(f)\,\rho^{k+1}. \tag{12}$$

With the first inequality of (6), and the inequality of (9′), we can write this as

$$M_f\left(\frac{\rho^2-1}{2\rho}\right) \leqslant \left\{|p_0(z)| + 2\sum_{k<n_0(\epsilon)} E_k(f)\,\rho^{k+1}\right\} + 2\sum_{k<n_0(\epsilon)} \frac{\rho^{k+1}}{k^{k/(\beta+\epsilon)}}. \tag{13}$$

It is known [*2*, p. 9] that

$$g(z) = \sum_{k=0}^{\infty} b_k z^k$$

is an entire function of (finite) order α if and only if

$$\overline{\lim_{n\to\infty}} \frac{n \ln n}{\ln(1/|b_n|)} = \alpha. \tag{14}$$

Applying this test to the last sum of (13), we see that this sum is an entire function of order $\beta + \epsilon$. Thus, there exists an $R(\epsilon) \geqslant 1$ such that

$$M_f\left(\frac{\rho^2-1}{2\rho}\right) \leqslant P_{n_0}(\rho) + \exp(\rho^{(\beta+2\epsilon)}) \qquad \text{for all } \rho > R(\epsilon), \tag{15}$$

where

$$P_{n_0}(\rho) \equiv |p_0(z)| + 2\sum_{k<n_0(\epsilon)} E_k(f)\,\rho^{k+1}$$

is a polynomial of degree at most $n_0(\epsilon)$. From (15), it then readily follows that

$$\overline{\lim_{\rho \to \infty}} \frac{\ln(\ln M_f(\rho))}{\ln \rho} \leqslant \beta, \tag{16}$$

which shows that the entire function $f(z)$ is of order at most β. Summarizing, if $f(z)$ is of (finite) order σ, then (9) is valid for some β with $\beta \leqslant \sigma$. If $\beta < \sigma$, the argument above leading to (16) shows that $f(z)$ would be of order less than σ, a contradiction. Thus, $\beta = \sigma$, and the circle of reasoning is complete for the converse as well. Q.E.D.

Finally, we remark that an even sharper result about the rate at which $E_n^{1/n}(f)$ tends to zero in (2) is possible under the assumption of smoother growth properties for $f(z)$. More precisely, an entire function $f(z)$ of positive order ρ is said [*2*, p. 8] to be of *type* τ $(0 \leqslant \tau < \infty)$ if

$$\overline{\lim_{r \to \infty}} \frac{\ln M_f(r)}{r^\rho} = \tau, \tag{17}$$

where

$$M_f(r) \equiv \max_{|z| \leqslant r} |f(z)|.$$

The following analogue of Theorem 1 is due in fact to Bernstein [*1*, p. 114], and can be proved in the manner of Theorem 1.

THEOREM 2. *Let* $f(x)$ *be a real-valued continuous function on* $[-1,+1]$. *Then, there exist constants* Λ (*positive*) *and* α, β (*nonnegative*) *such that*

$$\overline{\lim_{n \to \infty}} \{n^{1/\Lambda} E_n^{1/n}(f)\} = \alpha \tag{18}$$

if and only if $f(x)$ *has an analytic extension* $f(z)$ *such that* $f(z)$ *is an entire function of order* Λ *and type* β.

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